Micromaser theory in manifest Lindblad form

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Abstract. We discuss the laser theory for a single-mode micromaser that is pumped with a dilute stream of excited two-level atoms. In the weak-coupling regime, an expansion in the coupling strength is developed that preserves the Lindblad form of the master equation. This expansion breaks rapidly down above threshold. This can be improved with an alternative approach, not restricted to weak coupling: the Lindblad operators are expanded in orthogonal polynomials adapted to the probability distribution for the atom-laser interaction time. Results for the photon statistics and the laser linewidth illustrate the theory.
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1. Introduction

The quantum theory of a laser is a textbook example of a nonlinear problem that requires techniques from open quantum systems. The key issue is the nonlinearity in the gain of the laser medium, due to saturation, that leads to coupled nonlinear equations already at the semiclassical level. The quantum theory makes things worse by its use of non-commuting operators.

Recall that in the so-called semiclassical theory, the following equation of motion for the intensity I the laser mode can be derived [1, 2]:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\kappa I + \frac{AI}{1+\beta I} \tag{1}$$

where κ is the loss rate, A is the linear gain, and β describes gain saturation for the laser medium. A quantum upgrade of this theory replaces the intensity by the photon number $a^\dagger a$ where the annihilation operator a describes the field amplitude of the laser mode. Mode loss is easy to handle by coupling the laser mode linearly to a mode continuum 'outside' the laser cavity [3]. This leads to a master equation for the density matrix in so-called Lindblad form [see Eq.(6)] with a Lindblad operator $L = \sqrt{\kappa}a$. Linear gain can be handled in the same way, but gain saturation is more tricky. A heuristic conjecture is a Lindblad operator $L = \sqrt{A}\, a^\dagger (1 + B a^\dagger a)^{-1/2}$. The operator ordering can only be ascertained *a posteriori*, and it is difficult to choose among the replacements $I \mapsto a^\dagger a$, $aa^\dagger$, or $\frac{1}{2}\{a^\dagger a + aa^\dagger\}$.

In Refs.[2, 4], a simple approach to nonlinear gain is presented, based on a stream of two-level atoms that crosses the laser cavity and that are prepared in their excited state (with some probability). One gets a reduced density matrix for the laser mode by tracing over the atoms that leave the cavity. This problem can be largely handled exactly [5], even in the presence of incoherent effects like cavity damping, atom pumping, and frequency-shifting collisions. The setup has become known as the 'micromaser' because of its experimental realization with a high-quality cavity [6, 7, 8]. One line of research has focused on the so-called 'strong coupling regime' that permits the laser mode to be driven into non-classical states [9, 10].

We focus here on the 'weak coupling' regime. On the level of the master equation for the laser mode, this regime corresponds to a small product of coupling constant and elementary interaction time τ so that one can expand in this parameter. For the description of a realistic experiment, one has to average the master equation with respect to a distribution in τ (Sec. 2). It turns out, however, that the resulting master equation is not of the well-known Lindblad form, although it preserves the trace of the density matrix. In this paper, we give a discussion of this problem and suggest a solution. On the way, we review the derivation of the Lindblad master equation starting from the Kraus-Stinespring representation of the finite-time evolution of the density matrix (Sec. 3). The mathematical treatment is at the border of validity of the formal Lindblad theory since one has to deal with an infinite-dimensional Hilbert space and continuous sets of Kraus and Lindblad operators. We construct two modified expansions that result both in a Lindblad master equation (Secs. 4, 5). One is a direct amendment of the weak coupling approximation, the other one is able to enter the regime of a strong coupling (on average). The latter gives at least qualitative agreement with the results of the exact master equation.

2. The micromaser model

Consider a two-level atom with states $|g\rangle$, $|e\rangle$ that is prepared at time t in its excited state $|e\rangle = (1, 0)^T$ (density matrix $\rho_A = |e\rangle\langle e|$) and that interacts with a single mode (density matrix ρ) during a time τ. One adopts a Jaynes-Cummings-Paul Hamiltonian for the atom-field coupling

$$H_{\rm JCP} = \hbar g \left(a^\dagger \sigma + a \sigma^\dagger \right), \qquad \sigma = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

(this applies at resonance in a suitable interaction picture). Assume that the initial density operator of the atom+field-system factorizes into $P(t) = \rho(t) \otimes \rho_A$, compute $P(t+\tau)$ by solving the Schrödinger equation and get the following reduced field density matrix [2, 4]

$$\rho(t+\tau) = \cos(g\tau\hat{\varphi})\rho(t)\cos(g\tau\hat{\varphi}) + (g\tau)^2 a^\dagger \operatorname{sinc}(g\tau\hat{\varphi})\rho(t)\operatorname{sinc}(g\tau\hat{\varphi})a \quad (3)$$

where $\operatorname{sinc}(x) \equiv \sin(x)/x$, and $\hat{\varphi}^2 = aa^\dagger$ is one plus the photon number operator. The operator-valued functions $\cos$ and sinc are defined by their series expansion. Only even powers of the argument occur, hence we actually never face the square root $\hat{\varphi}$ of the operator $aa^\dagger$. In the following, we abbreviate the mapping defined by Eq.(3) by $\mathbb{M}_\tau\rho(t)$ (this is sometimes called a superoperator).

The operation (3) describes an elementary 'pumping event' of the laser. To provide a more realistic description, one introduces the following additional averages: a dilute stream of atoms is crossing the laser cavity, and feeds in excited atoms at a rate r such that $r\tau \ll 1$. The interaction time τ is itself distributed according to the probability measure $\mathrm{d}p(\tau)$ with mean value $\bar{\tau}$. On a coarse-grained time scale $\Delta t \gg \bar{\tau}$, this leads to the difference equation [2, 4]

$$\frac{\Delta\rho}{\Delta t} = r \int \mathrm{d}p(\tau)\ (\mathbb{M}_\tau - \mathbb{1})\,\rho. \quad (4)$$

To simplify the superoperator appearing on the right hand side, Refs.[2, 4] suggest an expansion in powers of $g\tau\hat{\varphi}$ up to the fourth order. Using an exponential distribution for $\mathrm{d}p(\tau)$, this leads to the approximate master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= A\left(a^\dagger\rho a - \tfrac{1}{2}\{aa^\dagger,\,\rho\}\right) \\ &\quad + \mathcal{B}\left(3aa^\dagger\rho aa^\dagger + \tfrac{1}{2}\{(aa^\dagger)^2,\,\rho\} - 2a^\dagger\{aa^\dagger,\,\rho\}a\right) \quad (5)\end{aligned}$$

where we followed the common practice of interpreting this as a differential equation. We use $\{\cdot,\cdot\}$ to denote the anticommutator. The linear gain is $A = 2r(g\bar{\tau})^2$, and $\mathcal{B} = (g\bar{\tau})^2 A$ is a measure of gain saturation. Losses from the laser mode can be included in the usual way by adding a term of the same structure as the first line of Eq.(5), but exchanging a and $a^\dagger$ and replacing A by the cavity decay rate κ [2, 4].

It is easy to check that Eq.(5) preserves the trace of ρ, using cyclic permutations. Nevertheless, it is not of the general form introduced by Lindblad [11]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \sum_\lambda \left(L_\lambda\rho L_\lambda^\dagger - \tfrac{1}{2}\left\{L_\lambda^\dagger L_\lambda,\,\rho\right\}\right) \quad (6)$$

with a countable set of operators L_λ. One may think of an *ansatz* polynomial in the a and $a^\dagger$ for the L_λ, but it is difficult to see how to generate the mixed third order-first order terms $a^\dagger aa^\dagger\rho a$ without generating also contributions like $a^\dagger aa^\dagger\rho aa^\dagger a$. Note that this 'missing term' cannot disappear by cancellations: if we allow the L_λ operators to contain at maximum three factors of a or $a^\dagger$, then the highest order term generated by the 'sandwich' structure $L_\lambda\rho L_\lambda^\dagger$ is proportional to the squared coefficient of the highest order term of L_λ, and these terms cannot cancel out.

Of course, one can accept to work with this kind of 'post-Lindblad' master equations (as they appear frequently in the papers of Golubev and co-workers, see e.g. [12, 13]). We follow here another route and raise the question: what assumptions behind the standard Lindblad master equation do not apply here, or are we missing something? To formulate an answer, we go back to a derivation of the Lindblad form that starts from another general formulation for mappings between density matrices, the so-called Kraus or Stinespring representation of completely positive operators [11]. We show that a set of Lindblad operators $\{L_\lambda\}$ can indeed be constructed so that by adding a few additional terms to the master equation (5), it can be brought into the Lindblad form.

3. Lindblad from Kraus–Stinespring

The time evolution of a density matrix can be expected to yield a density matrix again. This intuitively obvious requirement is violated by some models [14] or for some initial states [15], but it can also be taken as a starting point for an 'axiomatic' theory of dissipative quantum dynamics. Following this latter approach, one derives that the evolution over a finite time Δt must be of the form (Stinespring theorem, Ref.[11])

$$\rho(t + \Delta t) = \sum_\lambda \Omega_\lambda \rho(t) \Omega_\lambda^\dagger \tag{7}$$

where the operators Ω_λ depend on Δt and satisfy the 'completeness relation' $\sum_\lambda \Omega_\lambda^\dagger \Omega_\lambda = \mathbb{1}$ to ensure trace conservation. This form (called Kraus representation [11, 16]) can be easily secured for the micromaser master equation (4). We resolve the discrete difference quotient and get ($\lambda = 0, 1, 2$)

$$\Omega_0 = (1 - r\Delta t)^{1/2} \mathbb{1} \tag{8}$$

$$\Omega_1 = (r\Delta t)^{1/2} \cos(g\tau\hat{\varphi}) \tag{9}$$

$$\Omega_2 = (r\Delta t)^{1/2} g\tau a^\dagger \operatorname{sinc}(g\tau\hat{\varphi}) \tag{10}$$

where the completeness relation is satisfied because of the trigonometric identity $\sin^2 + \cos^2 = 1$ that is carried over to operator-valued arguments.

We note that the Kraus representation retains its form, at least formally, when we average the operators Ω_λ with respect to a distribution in the parameter τ. This is easily seen by interpreting the integral over τ as a Riemann sum: for each λ, the term $\Omega_\lambda(\tau)\rho\Omega_\lambda^\dagger(\tau)$ is replaced by the sum

$$\sum_j \Omega_{\lambda j} \rho \Omega_{\lambda j}^\dagger \quad \text{with} \quad \Omega_{\lambda j} \equiv \Omega_\lambda(\tau_j) \sqrt{\mathrm{d}p(\tau_j)} \tag{11}$$

The completeness relation is also still satisfied: for each τ_j, $\Omega_1^\dagger(\tau_j)\Omega_1(\tau_j)$ and $\Omega_2^\dagger(\tau_j)\Omega_2(\tau_j)$ still add up to the unit operator, and the sum over the prefactors goes over into the normalization integral of the probability measure $\mathrm{d}p(\tau)$. Note that we interchange here the summations over λ and j.

The derivation of the Lindblad master equation (see, for example, Ref.[16] and the Appendix) now provides a construction of the operators L_λ appearing in Eq.(6). Extract a traceless operator V_λ by writing

$$\Omega_\lambda = \omega_\lambda \mathbb{1} + V_\lambda \tag{12}$$

and take the limit

$$L_\lambda = \lim_{\Delta t \to 0} \frac{V_\lambda}{\sqrt{\Delta t}}. \tag{13}$$

The operator Ω_0 is already proportional to the unit matrix, hence $V_0 = 0$. It is also obvious that Ω_2 is traceless, hence

$$L_2 = S \equiv \sqrt{r}\, g\tau a^\dagger \operatorname{sinc}(g\tau\hat{\varphi}) \tag{14}$$

The operator Ω_1 has a singular trace. In the number state basis:

$$\sum_{n=0}^{\infty} \langle n|\Omega_1(\tau)|n\rangle = (r\Delta t)^{1/2} \sum_{n=0}^{\infty} \cos(g\tau\sqrt{n+1}) \tag{15}$$

The subset of square numbers gives a divergent result whenever $g\tau = 0$ modulo 2π, hence a comb of δ-functions is expected. This is of course a tricky result in view of the expansion in $g\tau$ that is operated in the way from Eq.(4) to Eq.(5). We therefore introduce a factor q^n with $0 < q < 1$ into the sums (15). Comparing the traces of both sides in Eq.(12) for $\lambda = 1$ (using that V_1 is traceless), we get

$$\omega_1 = (r\Delta t)^{1/2} \varpi(g\tau) \equiv (r\Delta t)^{1/2} \sum_{n=0}^{\infty} (1-q) q^n \cos(g\tau\sqrt{n+1}). \tag{16}$$

Finally, we find

$$L_1 = C \equiv \sqrt{r}\, [\cos(g\tau\hat{\varphi}) - \varpi_1(g\tau)\mathbb{1}] \tag{17}$$

where $\varpi_1(g\tau)$ is defined in (16).

Observe that at this stage, we do get a master equation in Lindblad form. But the Lindblad operators still contain the interaction time $g\tau$ to all orders.

4. Weak-coupling Lindblad form

We now investigate how the expansion in powers of $g\tau$ and the averaging with respect to $\mathrm{d}p(\tau)$ can be organized so that the Lindblad form is preserved.

4.1. Consistency of the expansion

We start with two general remarks. Consider a polynomial approximation of order N to the $\cos$ and sinc functions in Eqs.(14, 17). The operator $C^\dagger C$ is then of order $2N$ in $g\tau$ and the operator $S^\dagger S$ of order $2N+2$ [cf. Eqs.(14,17)]. The maximum number of factors a or $a^\dagger$ in the master equation is given by $2N+2$. A scheme consistent with the Lindblad form thus seems possible only if the master equation is expanded at least to the order $2N+2$ in $g\tau$.

The case $2N+2 = 2$, involves the second order $(g\tau)^2$ only, hence a Lindblad-like form with rate coefficient $2r(g\bar{\tau})^2 = A$. This reproduces the first line of the master equation (5).

The next case is $2N+2 = 6$ because both $\cos$ and sinc are even in τ. Then one should have six factors a or $a^\dagger$. We see that this is not the case in Eq.(5) where only up to four factors appear. Therefore, the expansion in $g\tau\hat{\varphi}$ has not been pushed to a high enough order (sixth) to be compatible with a Lindblad form, at least for the terms originating from the Lindblad operator S.

A second remark: consider the expansion in powers of $g\tau$ of the operators C, S:

$$C = \sqrt{r} \sum_{n=0}^{\infty} (g\tau)^n C_n \tag{18}$$

$$S = \sqrt{r}\, a^\dagger \sum_{n=0}^{\infty} (g\tau)^n S_n \tag{19}$$

where the coefficients C_n, S_n involve powers of the operator $\hat{\varphi} = (aa^\dagger)^{1/2}$. The integration over τ now leads to 'cross terms' like

$$\int \mathrm{d}p(\tau)\, (g\tau)^{n+m} C_n \rho C_m. \tag{20}$$

The resulting master equation is not in diagonal form because once the integration over τ performed, these cross terms cannot be written as a product of a function of n times a function of m. Progress can be made by using an expansion in orthogonal polynomials, as we discuss now.

4.2. Polynomial expansion

In the expansions (18, 19) of the Liouville operators, let us re-write the powers as

$$\tau^n = \sum_{k=0}^{n} a_{nk} \bar{\tau}^n f_k(\tau/\bar{\tau}). \tag{21}$$

The factor $\bar{\tau}^n$ is chosen for dimensional convenience. In the example discussed below, $\bar{\tau}$ is identified with the mean value of the probability measure $\mathrm{d}p(\tau)$. The polynomials $f_k(\tau/\bar{\tau})$ are of order k and are orthogonal with respect to the following scalar product

$$\int \mathrm{d}p(\tau)\, f_k(\tau/\bar{\tau}) f_l(\tau/\bar{\tau}) = \delta_{kl}. \tag{22}$$

This is a scalar product since $\mathrm{d}p(\tau)$ is a positive measure. Such polynomials exist and are real. An explicit example is worked out below for an exponential distribution. The coefficients a_{nk} in Eq.(21) can be found by projecting x^n onto $f_k(x)$ which boils down to an integral similar to (22). Note that $a_{nk} = 0$ for $k > n$ because x^n can be written as a finite linear combination of the $f_l(x)$ $(0 \le l \le n)$, using the Gram-Schmidt procedure for orthogonalization. We then get for the first cross term in (20):

$$\int \mathrm{d}p(\tau)\, \tau^{n+m} = \bar{\tau}^{n+m} \sum_{k=0}^{\min(n,m)} a_{nk} a_{mk}. \tag{23}$$

The average of the master equation involving the Lindblad operator C, say, then assumes the following diagonal form

$$\begin{aligned}&\int \mathrm{d}p(\tau)\, \left(C\rho C^\dagger - \tfrac{1}{2} \left\{ C^\dagger C,\, \rho \right\} \right) \\ &= r \sum_{k=0}^{\infty} \sum_{n,m=k}^{\infty} a_{nk} a_{mk} (g\bar{\tau})^{n+m} \left(C_n \rho C_m^\dagger - \tfrac{1}{2} \left\{ C_m^\dagger C_n,\, \rho \right\} \right)\end{aligned} \tag{24}$$

which can be written in a Lindblad form involving the (countable set of) operators

$$\tilde{C}_k = \sqrt{r} \sum_{n=k}^{\infty} a_{nk} (g\bar{\tau})^n C_n. \tag{25}$$

If the expansion (19) is truncated at order N, then $\tilde{C}_k$ involves also only terms up to order $n = N$ and the set of Lindblad operators is finite as well. Let us consider $N = 2$ and take into account that C is even in τ. Then

$$\tilde{C}_0 = a_{00} C_0 + a_{20} (g\bar{\tau})^2 C_2 \tag{26}$$

$$\tilde{C}_1 = a_{21} (g\bar{\tau})^2 C_2 \tag{27}$$

$$\tilde{C}_2 = a_{22} (g\bar{\tau})^2 C_2 \tag{28}$$

The Lindblad operators $\tilde{C}_{1,2}$ can be combined into a single one since they are proportional to the same operator C_2. A similar procedure can be applied to S, the only difference being that only odd coefficients S_1, S_3, ... are nonzero.

4.3. Example: Laguerre polynomials

The Laguerre polynomials $L_n(x)$ implement orthogonality with respect to a scalar product weighted with an exponential [17]

$$\int_0^\infty \mathrm{d}x \, \mathrm{e}^{-x} L_n(x) L_m(x) \propto \delta_{nm} \tag{29}$$

which corresponds to the probability distribution $\mathrm{d}p(\tau) = (\mathrm{d}\tau/\bar{\tau})\mathrm{e}^{-\tau/\bar{\tau}}$ considered by Orszag [2]. We identify $x = \tau/\bar{\tau}$ as the natural variable for the polynomials we require. The first few Laguerre polynomials read, normalized as in Eq.(22)

$$\begin{aligned} f_0(x) &= 1 \qquad\qquad f_1(x) = 1 - x \\ f_2(x) &= \frac{1}{2}\left(x^2 - 4x + 2\right) \end{aligned} \tag{30}$$

A straightforward calculation gives the following Lindblad operators for the master equation (5):

$$\tilde{C}_0 = \sqrt{r}(g\bar{\tau})^2 \left(\frac{\mathbb{1}}{(1-q)} - aa^\dagger \right), \quad \tilde{C}_1 = -2\tilde{C}_0, \quad \tilde{C}_2 = \tilde{C}_0 \tag{31}$$

$$\tilde{S}_0 = \sqrt{r} g\bar{\tau} a^\dagger \left(\mathbb{1} - (g\bar{\tau})^2 aa^\dagger \right) \tag{32}$$

$$\tilde{S}_1 = -\sqrt{r} g\bar{\tau} a^\dagger \left(\mathbb{1} - 3(g\bar{\tau})^2 aa^\dagger \right) \tag{33}$$

$$\tilde{S}_2 = \sqrt{10\, r}(g\bar{\tau})^3 a^\dagger a a^\dagger \tag{34}$$

The operators $\tilde{C}_{0,1,2}$ are proportional to each other and can be combined into a single one (replace $\sqrt{r}$ by $\sqrt{6\, r}$ in $\tilde{C}_0$). An analoguous simplification has been already made in writing Eq.(34). Working out the details, we see that the part of $\tilde{C}_0$ that involves $\mathbb{1}/(1-q)$ actually does not contribute to the master equation. (This is generally true if we have a hermitean Lindblad operator and add a term proportional to the unit operator with a real coefficient.)

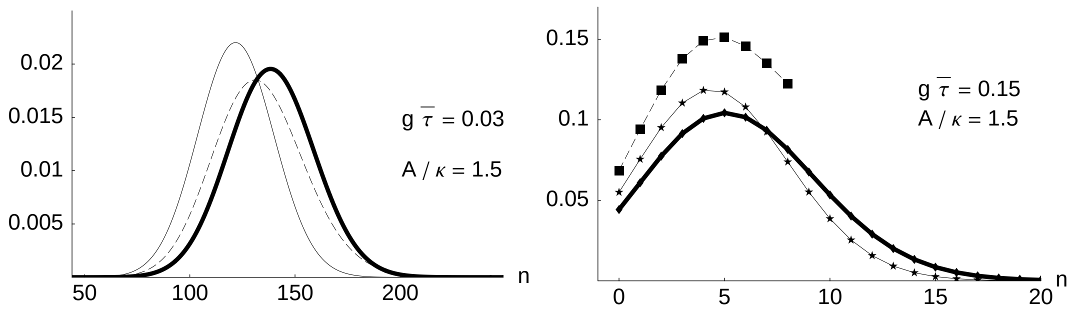

Figure 1. Equilibrium photon statistics for the micromaser. Thick solid line: exact theory (4), as worked out in Ref.[2]. Dashed line: Lindblad theory in the weak coupling limit with operators (31–34). Thin solid line: Lindblad theory in the uniform approximation (38–40), see Section 5. The average interaction time is fixed to $g\bar{\tau} = 0.03$ (left) and $g\bar{\tau} = 0.15$ (right).

We can now identify the 'missing pieces' in the master equation (5). Collecting the third-order terms arising from $\tilde{S}_{0,1,2}$ gives

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{6\text{th}} = 20\, r (g\bar{\tau})^6 \left(a^\dagger a a^\dagger \,\rho\, a a^\dagger a - \tfrac{1}{2}\{(a a^\dagger)^3,\, \rho\} \right). \tag{35}$$

These terms are, of course, of sixth order in $(g\bar{\tau})^6$ and, not really surprisingly, themselves in Lindblad form. All other terms are of lower order in $g\bar{\tau}$ and combine to reproduce Eq.(5).

4.4. Numerical results

To illustrate the accuracy of the expansion performed here, we have worked out the equilibrium photon statistics p_n, i.e., the diagonal elements $\langle n|\rho_{\mathrm{eq}}|n\rangle$ of the stationary solution to the master equation. Two examples are shown in Fig. 1, for the same value of the pumping parameter A/κ and different values of the coupling strength $g\bar{\tau}$. The photon statistics is fairly well approximated at weak coupling ($g\bar{\tau} = 0.03$), as expected. At the value $g\bar{\tau} = 0.15$, the average photon number is not very large, and significant differences occur. One has to face the technical difficulty that the recurrence relation for the photon statistics (see Refs.[1, 2]) diverges in the weak-coupling approximation for $n \gg n_{\mathrm{cut}} = \frac{1}{5}(g\bar{\tau})^{-2}$. This does not change the results if this number is well beyond the peak of p_n (weak coupling). But as $g\bar{\tau}$ increases, the probabilities p_n ($n \approx n_{\mathrm{cut}}$) near the cutoff are still significant, and the approximation breaks down.

The average photon number $\langle n\rangle$ and its normalized variance $Q = (\Delta n)^2/\langle n\rangle$ (essentially the so-called Mandel parameter) are plotted in Figs. 2 and 3. A similar trend can be observed, with the weak coupling expansion giving an accurate description below and slightly above threshold. The agreement is the better, the smaller the coupling parameter $g\bar{\tau}$. Above threshold, the expansion is no longer useful because photon numbers with $g\bar{\tau}\sqrt{n+1} \sim 1$ are significantly populated. At threshold, the photon number fluctuations are strongly super-Poissonian (the Mandel parameter

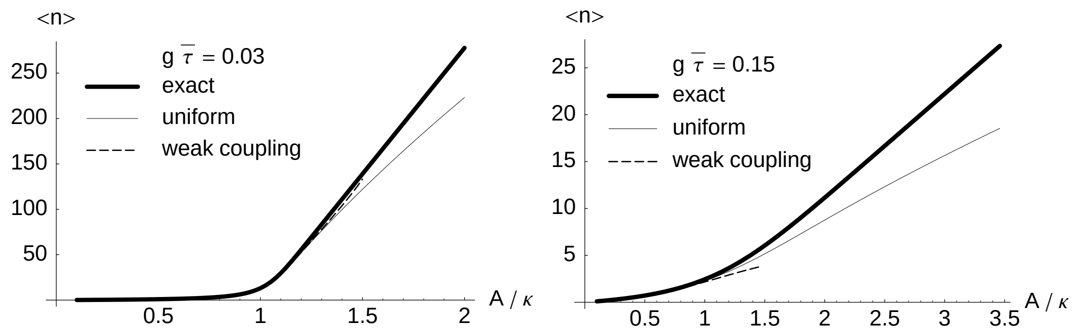

Figure 2. Average photon number (i.e., laser output intensity) vs. pumping strength. Left: weak coupling, $g\bar{\tau} = 0.03$; right: stronger coupling $g\bar{\tau} = 0.15$.
Thick solid line: exact theory; dashed line: Lindblad theory for weak coupling; thin solid line: uniform Lindblad theory in the uniform.

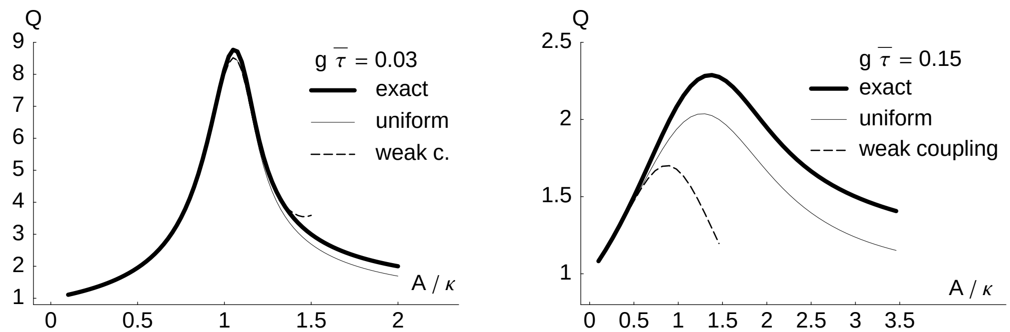

Figure 3. Normalized variance of the photon number, $Q = (\Delta n)^2/\langle n\rangle$ (Mandel parameter), vs. pumping strength. The curves are labelled as in Fig.2.

$Q > 1$). They tend to the Poisson (or coherent state) limit above threshold, but this regime is not accessible with the weak coupling expansion. We develop an alternative description (leading to the thin solid lines) in the following Section.

Finally, we plot in Fig.4 the following estimate for the laser linewidth

$$D = -\frac{2}{\langle n\rangle}\left\langle \frac{\mathrm{d}a^\dagger(t+\tau)}{\mathrm{d}\tau} a(t)\right\rangle_{\tau\to 0} \tag{36}$$

where the derivative with respect to τ is evaluated using the master equation and we consider $t \to \infty$ so that the expectation value is taken with respect to the stationary state. The data plotted in the Figure are normalized with respect to $\kappa/\langle n\rangle$ which is of the order of the Schawlow-Townes linewidth. Values close to one indicate the line narrowing typical for a laser above threshold. We see that the weak coupling approximation rapidly deviates above threshold. At strong coupling, significant deviations from the Schawlow-Townes limit occur in all descriptions. This can be traced back to an additional, positive contribution from to the C-operators in the master equation. Note that in both the exact and approximated theory, these operators are diagonal in the number state basis and hence do not influence the photon statistics. We shall report on a more detailed analysis elsewhere.

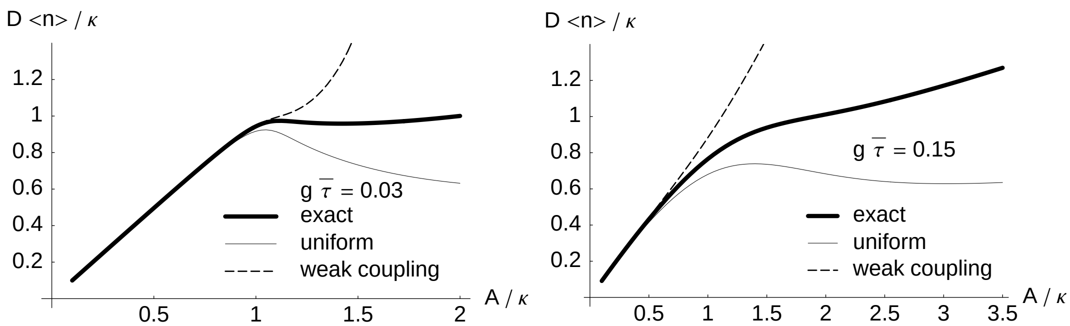

Figure 4. Normalized linewidth D of the micromaser, vs. the pumping strength. The linewidth is normalized to the Schawlow-Townes limit $\kappa/\langle n\rangle$. The curves are labelled as in Fig.2.

5. Uniform expansion

To conclude, we discuss an alternative expansion for the Lindblad operators. The idea is to perform an expansion of the operators $C(g\tau\hat{\varphi})$ and $S(g\tau\hat{\varphi})$ in Laguerre polynomials in τ. The average with respect to $\mathrm{d}p(\tau)$ is then easy due to the orthogonality relation (29). This gives a different dependence on $\hat{\varphi}$ and $g\bar{\tau}$ where actually the operator $\hat{\varphi}$ appears to all orders. We shall see that this approximation provides a convergent photon statistics even above threshold.

For simplicity, we retain in the expansion only the lowest order polynomials and approximate the operator S by $f_0(\tau/\bar{\tau})S_{0,\mathrm{uniform}} + f_1(\tau/\bar{\tau})S_{1,\mathrm{uniform}}$. We have

$$S_{k,\mathrm{uniform}} = \int_0^\infty \mathrm{d}p(\tau)\, S(g\tau\hat{\varphi}) f_k(\tau/\bar{\tau}), \quad k = 0, 1, \tag{37}$$

which results in

$$\tilde{S}_{0,\mathrm{uniform}} = \sqrt{r}\, g\bar{\tau}\, a^\dagger \frac{1}{1 + (g\bar{\tau})^2 a a^\dagger}, \tag{38}$$

$$\tilde{S}_{1,\mathrm{uniform}} = -\sqrt{r}\, g\bar{\tau}\, a^\dagger \frac{1 - (g\bar{\tau})^2 a a^\dagger}{[1 + (g\bar{\tau})^2 a a^\dagger]^2}. \tag{39}$$

Note that both operators contribute at order $\sqrt{A}$ for small photon numbers such that $(g\bar{\tau})^2(n+1) \ll 1$. The reduction of the weak signal gain occurs through the denominators that involve the photon number, similar to the Scully-Lamb laser theory [1]. But observe that gain saturation even happens with the laser mode in the vacuum state. This regime has been studied previously to prepare, e.g., non-classical states in the micromaser. In the visible frequency band, the regime is accessible for microlasers with high-Q cavities [18, 19].

A similar calculation leads to

$$\tilde{C}_{0,\mathrm{uniform}} = \frac{\sqrt{r}}{1 + (g\bar{\tau})^2 a a^\dagger} - \mathbb{1} \sum_{n=0}^{\infty} \frac{\sqrt{r}\,(1-q)q^n}{1 + (g\bar{\tau})^2(n+1)} \tag{40}$$

which features the same gain saturation. Observe again that the part proportional to the unit operator actually drops out from the master equation. The following orders (involving the polynomials $f_{1,2}(x)$) are proportional to $\sqrt{r}(g\bar{\tau})^2 = \sqrt{A}(g\bar{\tau})$ and can be included to systematically improve the approximation.

The matrix elements of these 'uniform' operators are decreasing as the photon number gets large. This provides a recurrence relation for the photon statistics that converges for $n \to \infty$. We indeed observe from the numerical results shown in Figs. 1–3 that the divergences of the weak coupling approximation are removed. The behaviour of the exact theory for all considered observables is well reproduced. We speculate that additional terms in the Laguerre expansion will improve the results for the laser linewidth (Fig. 4) where the agreement is worse than for the moments of the photon statistics (Figs. 2, 3).

Let us finally comment on the heuristic choice $L = \sqrt{A}a^\dagger(1+Ba^\dagger a)^{-1/2}$ mentioned in the Introduction. The choice $B = 4(g\bar{\tau})^2$ leads, by construction, to the same photon statistics as the exact theory. It does not feature the onset of gain saturation already for the vacuum state (as the micromaser theory does), unless one changes the order of operators. Another shortcoming is the laser linewidth that is not correctly reproduced, as additional contributions arise from the C-type Lindblad operators. Hence, for the micromaser at hand, this approximation is not suitable. It can be used as an introductory tool for more conventional lasers, with the advantage that one automatically gets a master equation that is trace preserving and whose rate equations satisfy detailed balance.

6. Conclusion

A dilute jet of excited two-level atoms that crosses a maser or laser cavity provides a nontrivial example for a laser with nonlinear gain. It has been studied both experimentally and theoretically for a long time already. We have pointed out here that an expansion of the master equation in the weak coupling regime can be organized in such a way that the equation retains its Lindblad form explicitly. The expansion is based on polynomials that are orthogonal with respect to the probability distribution of the atomic transit time through the laser mode. An alternative scheme that is able to handle the strong coupling regime as well has been suggested and leads to a reasonable agreement with the exact theory. Further work will address a detailed analysis of the laser linewidth and the strong coupling regime.

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Appendix A. Derivation of the Lindblad form

In an axiomatic approach to the time evolution of density matrices, it can be shown that over a time Δt, the density matrix changes according to Eq.(7) [11, 16]. The

Lindblad theorem then states:

> *Suppose that the time evolved density operator has the weak continuity property*
>
> $$\lim_{\Delta t\to 0}\left[\hat{A}\rho(t+\Delta t)-\hat{A}\rho(t)\right]=\mathcal{O}(\Delta t) \tag{A.1}$$
>
> *for all operators $\hat{A}$ and initial density matrices $\rho(t)$. Then there exists a hermitean operator H and a set of traceless operators L_λ such that*
>
> $$\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\mathrm{i}\,[H,\,\rho]+\sum_\lambda\left(L_\lambda^\dagger\rho L_\lambda-\tfrac{1}{2}\{L_\lambda L_\lambda^\dagger,\,\rho\}\right) \tag{A.2}$$
>
> *This differential equation is called the Lindblad form and the L_λ are called Lindblad operators.*

Proof. Let $\Delta t>0$ and write $\rho=\rho(t)$ for simplicity. We start with the Kraus representation (7) for the density matrix $\rho(t+\Delta t)$,

$$\rho(t+\Delta t)=\sum_\lambda\Omega_\lambda\rho\Omega_\lambda^\dagger \tag{A.3}$$

The operators occurring in Eq.(A.3) can be split into

$$\Omega_\lambda=\omega_\lambda\mathbb{1}+V_\lambda \tag{A.4}$$

where the V_λ are uniquely defined by the requirement that their trace be zero. Note that ω_λ and V_λ depend in general on Δt.

In terms of these quantities, the change in the density matrix is computed to be

$$\begin{aligned}\rho(t+\Delta t)-\rho=&\left(\sum_\lambda|\omega_\lambda|^2-1\right)\rho+\sum_\lambda\left(\omega_\lambda^*V_\lambda\,\rho+\rho\,\omega_\lambda V_\lambda^\dagger\right)\\&+\sum_\lambda V_\lambda^\dagger\rho V_\lambda\end{aligned} \tag{A.5}$$

where ω_λ^* is complex conjugate to ω_λ. Using the continuity condition (A.1) for all operators $\hat{A}$ and ρ, we find

$$\lim_{\Delta t\to 0}\sum_\lambda|\omega_\lambda|^2=1 \tag{A.6}$$

$$\lim_{\Delta t\to 0}\sum_\lambda\omega_\lambda^*V_\lambda=0 \tag{A.7}$$

$$\lim_{\Delta t\to 0}\sum_\lambda V_\lambda\,\rho V_\lambda^\dagger=0 \tag{A.8}$$

where the last line applies to any density matrix ρ. We can thus introduce the derivatives

$$\gamma\equiv\lim_{\Delta t\to 0}\frac{\sum_\lambda|\omega_\lambda|^2-1}{\Delta t} \tag{A.9}$$

$$\Gamma-iH\equiv\lim_{\Delta t\to 0}\frac{\sum_\lambda\omega_\lambda^*V_\lambda}{\Delta t} \tag{A.10}$$

where Γ and H are both hermitean.

Differentiating the condition that the dynamical map preserves the trace of the density matrix, we find

$$\begin{aligned} 0 &= \lim_{\Delta t \to 0} \frac{\mathrm{tr}\left[\rho(t+\Delta t) - \rho\right]}{\Delta t} \\ &= \mathrm{tr}\Big[\gamma\rho + 2\Gamma\rho + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_\lambda V_\lambda^\dagger V_\lambda \, \rho\Big] \end{aligned} \tag{A.11}$$

Since this must hold for any density matrix ρ, we find another derivative

$$\lim_{\Delta t \to 0} \frac{\sum_\lambda V_\lambda^\dagger V_\lambda}{\Delta t} = -\gamma - 2\Gamma \tag{A.12}$$

We can thus introduce the Lindblad operators L_λ by the limiting procedure

$$L_\lambda \equiv \lim_{\Delta t \to 0} \frac{V_\lambda}{\sqrt{\Delta t}} \tag{A.13}$$

Using the derivatives defined in Eqs.(A.9, A.10, A.13), we can divide the difference $\rho(t+\Delta t) - \rho(t)$ in Eq.(A.5) by Δt, and take the limit $\Delta t \to 0$. This gives the differential equation (A.2). ■

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